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Uniform coverings of 2-paths with 5-paths in the complete graph☆

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Abstract

Let $n \geqslant 6$. There exists a uniform covering of 2-paths with 5-paths in K_n if and only if n is even or $n \equiv 1 \pmod 8$.

Keywords: 2-path; Covering; Uniform

1. Introduction

Let K_n be the complete graph on n vertices. A *path* of *length* l, or an l-*path*, is the graph induced by the edges in $\{\{v_i, v_{i+1}\} \mid 0 \leqslant i \leqslant l-1\}$; it is denoted by $[v_0, v_1, \ldots, v_l]$.

A uniform covering of the 2-paths in K_n with l-paths [l-cycles] is a set S of l-paths [l-cycles] having the property that each 2-path in K_n lies in exactly one l-path [l-cycle] in S. Only the following cases of the problem of constructing a uniform covering of the 2-paths in K_n with l-paths or l-cycles have been solved:

1. with 3-cycles,
2. with 3-paths [1],
3. with 4-cycles [2],
4. with 4-paths [4],

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5. with 5-paths when n is even [5],
6. with n-cycles (Hamilton cycles) when n is even [3].

In this paper, we solve the problem in the case of 5-paths, that is, we prove:

Theorem 1.1. *Let* $n \geqslant 6$. *There exists a uniform covering of* 2-*paths with* 5-*paths in* K_n *if and only if* n *is even or* $n \equiv 1 \pmod 8$.

2. Notation and preliminaries

Proposition 2.1. *There exists a uniform covering of* 2-*paths with* 5-*paths in* K_9.

Proof. Let $V_9 = \{0, 1, 2, \ldots, 8\}$ be the vertex set of K_9. Let σ_9 be the vertex-rotation $(0\ 1\ 2\ \cdots\ 8)$. Put $P_1 = [1, 8, 3, 6, 5, 4]$, $P_2 = [1, 8, 4, 5, 7, 2]$, $P_3 = [8, 1, 2, 7, 6, 3]$, $P_4 = [7, 2, 6, 3, 5, 4]$, $P_5 = [0, 2, 7, 4, 5, 8]$, $P_6 = [0, 6, 3, 1, 8, 5]$, $P_7 = [0, 8, 1, 7, 2, 3]$. Then $\{\sigma_9^i P_j \mid 0 \leqslant i \leqslant 8, 1 \leqslant j \leqslant 7\}$ is a uniform covering of 2-paths with 5-paths in K_9. $\square$

Proposition 2.2 (*Kobayashi* [5]). *Let* $n \geqslant 6$ *be even. There exists a uniform covering of* 2-*paths with* 5-*paths in* K_n.

Let $k \geqslant 1$. From Propositions 2.1 and 2.2, there exist uniform coverings of 2-paths with 5-paths in K_{8k} and K_9. Let $\mathscr{C}_1, \mathscr{C}_2$ be the coverings, respectively. Put $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$. We will construct a uniform covering of 2-paths with 5-paths in K_{8k+9}. We use $\mathscr{C}$ as a subset of the uniform covering in K_{8k+9}.

Let $K_{8k} = (V_1, E_1)$, $V_1 = \{0, 1, \ldots, 8k-1\}$, $K_9 = (V_2, E_2)$, $V_2 = \{a, b, c, d, e, f, g, h, i\}$, and $K_{8k+9} = (V, E)$, where $V = V_1 \cup V_2$, $V_1 \cap V_2 \neq \phi$. The set of 2-paths in K_{8k+9} that do not belong to $\mathscr{C}$ is $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$, where,

$$\Pi_1 = \{[x, t, y] \mid x, y \in V_1, t \in V_2\},$$

$$\Pi_2 = \{[x, y, t] \mid x, y \in V_1, t \in V_2\},$$

$$\Pi_3 = \{[t, x, u] \mid x \in V_1, t, u \in V_2\},$$

$$\Pi_4 = \{[t, u, x] \mid x \in V_1, t, u \in V_2\}.$$

We denote by σ the vertex-rotation in K_{8k}: $\sigma = (0\ 1\ 2\ \cdots\ 8k-1)$. We denote by τ the vertex-rotation in K_9: $\tau = (a\ b\ c\ d\ e\ f\ g\ h\ i)$. We denote by $\lambda_1, \lambda_2, \ldots, \lambda_8$ the vertex-permutations in K_9:

$$\lambda_1 : \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \longrightarrow \begin{pmatrix} b & c & a \\ e & f & d \\ h & i & g \end{pmatrix} \quad \lambda_2 : \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \longrightarrow \begin{pmatrix} c & a & b \\ f & d & e \\ i & g & h \end{pmatrix}$$

This notation means $\lambda_1(a)=b$, $\lambda_1(b)=c$, $\lambda_1(c)=a$, $\lambda_1(d)=e,\ldots,\lambda_1(i)=g$.

$$\lambda_3:\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\longrightarrow\begin{pmatrix}d&e&f\\g&h&i\\a&b&c\end{pmatrix}\quad \lambda_4:\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\longrightarrow\begin{pmatrix}e&f&d\\h&i&g\\b&c&a\end{pmatrix}$$

$$\lambda_5:\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\longrightarrow\begin{pmatrix}f&d&e\\i&g&h\\c&a&b\end{pmatrix}\quad \lambda_6:\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\longrightarrow\begin{pmatrix}g&h&i\\a&b&c\\d&e&f\end{pmatrix}$$

$$\lambda_7:\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\longrightarrow\begin{pmatrix}h&i&g\\b&c&a\\e&f&d\end{pmatrix}\quad \lambda_8:\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\longrightarrow\begin{pmatrix}i&g&h\\c&a&b\\f&d&e\end{pmatrix}$$

We define $\lambda_0=1$ (the identity permutation).

Let $A=\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}$. There are three rows $\{a,b,c\}$, $\{d,e,f\}$,$\{g,h,i\}$, three columns $\{a,d,g\}$, $\{b,e,h\}$, $\{c,f,i\}$, three right diagonals $\{a,e,i\}$, $\{d,h,c\}$, $\{g,b,f\}$ and three left diagonals $\{c,e,g\}$, $\{f,h,a\}$,$\{i,b,d\}$. We have the following proposition.

Proposition 2.3. *Let $t,u\in V_2$ $(t\neq u)$ such that t and u belong to the same row* [*column, right diagonal, left diagonal*] *of A. For any $v,w\in V_2$ $(v\neq w)$ such that v and w belong to the same row* [*column, right diagonal, left diagonal*], *there exists j $(0\leqslant j\leqslant 8)$ such that $\lambda_j(\{t,u\})=\{v,w\}$.*

Put $\Sigma=\{\sigma^j\,|\,0\leqslant j\leqslant 8k-1\}$, $\Sigma^*=\{\sigma^j\,|\,0\leqslant j\leqslant 4k-1\}$, $T=\{\tau^j\,|\,0\leqslant j\leqslant 8\}$ and $\Lambda=\{\lambda_j\,|\,0\leqslant j\leqslant 8\}$. If $\mathscr{P}$ is a set of paths in K_{8k+9} and Γ is a set of vertex-permutations in K_{8k+9}, we define $\Gamma\mathscr{P}=\{\gamma P\,|\,\gamma\in\Gamma,\ P\in\mathscr{P}\}$. A path Q is *contained in* a path P if Q is a subgraph of P. More generally, a path Q is *contained in* a set of paths $\mathscr{P}$ if Q is contained in one of the paths of $\mathscr{P}$. Define $\pi(\mathscr{P})=\{[x,y,z]\,|\,[x,y,z]$ is contained in $\mathscr{P}\}$.

For any edge $\{x,y\}$ in K_{8k}, we define the length $d(x,y)$:

$$d(x,y)=(y-x)\pmod{8k}.$$

For any two lengths d_1,d_2, we define that d_1 and d_2 are equal as lengths when $d_1=d_2$ or $d_1=-d_2\pmod{8k}$.

3. 5-paths of type A

Let l_1,l_2,l_3 be integers with $1\leqslant l_1,l_2,l_3<4k$. For a triplet $L=(l_1,l_2,l_3)$, we define P_L,Q_L,R_L in K_{8k+9} as follows:

$$P_L=[0,a,l_1,4k+l_2,b,4k],$$

$$Q_L=[c,-l_3,0,b,l_2,4k+l_1],$$

$$R_L=[c,4k-l_3,4k,a,4k+l_1,l_2].$$

Suppose $4k + l_1 \neq -l_3$ and $4k - l_3 \neq l_2 \pmod{8k}$, i.e., $l_1 + l_3 \neq 4k$ and $l_2 + l_3 \neq 4k \pmod{8k}$. Then P_L, Q_L, R_L are 5-paths in K_{8k+9}. We have the following proposition.

Proposition 3.1. *Let $L = (l_1, l_2, l_3)$ with $1 \leqslant l_1, l_2, l_3 < 4k$, $l_1 + l_3 \neq 4k$ and $l_2 + l_3 \neq 4k \pmod{8k}$. Then P_L, Q_L and R_L are 5-paths. Suppose $l_1 \neq l_2$, and l_3 and $l_1 - l_2 + 4k$ are not the same length. Then*

$$\pi(T\Sigma^*\{P_L, Q_L, R_L\}) = \{[x, t, y] \mid d(x, y) = l_1, l_2;\; x, y \in V_1, t \in V_2\}$$
$$\cup \{[x, y, t] \mid d(x, y) = l_3, l_1 - l_2 + 4k;\; x, y \in V_1, t \in V_2\}.$$

Proof. Let $x, y \in V_1$. The 2-path $[x, a, y]$ with $d(x, y) = l_1$ is contained in $\Sigma^*\{P_L, R_L\}$, so $[x, t, y]$ with $d(x, y) = l_1$ and $t \in V_2$ is contained in $T\Sigma^*\{P_L, R_L\}$. The 2-path $[x, b, y]$ with $d(x, y) = l_2$ is contained in $\Sigma^*\{P_L, Q_L\}$, so $[x, t, y]$ with $d(x, y) = l_2$ and $t \in V_2$ is contained in $T\Sigma^*\{P_L, Q_L\}$.

The 2-path $[x, y, a]$ with $y - x = l_1 - l_2 + 4k$ is contained in $\Sigma^*\{P_L, R_L\}$, so $[x, y, t]$ with $y - x = l_1 - l_2 + 4k$ and $t \in V_2$ is contained in $T\Sigma^*\{P_L, R_L\}$. (Addition is modulo $8k$ as $x, y \in V_1$.) The 2-path $[x, y, b]$ with $y - x = l_2 - l_1 + 4k$ is contained in $\Sigma^*\{P_L, Q_L\}$, so $[x, y, t]$ with $y - x = l_2 - l_1 + 4k$ and $t \in V_2$ is contained in $T\Sigma^*\{P_L, Q_L\}$. Hence $[x, y, t]$ with $d(x, y) = l_1 - l_2 + 4k$ and $t \in V_2$ is contained in $T\Sigma^*\{P_L, Q_L, R_L\}$. (Lengths $l_2 - l_1 + 4k$ and $l_1 - l_2 + 4k$ are the same length from the definition.)

The 3-paths $[c, -l_3, 0, b]$ and $[c, 4k - l_3, 4k, a]$ are contained in $\{Q_L, R_L\}$, so the 2-path $[x, y, t]$ with $d(x, y) = l_3$ and $t \in V_2$ is contained in $T\Sigma^*\{Q_L, R_L\}$.

Therefore $\pi(T\Sigma^*\{P_L, Q_L, R_L\}) \supset \{[x, t, y] \mid d(x, y) = l_1, l_2; x, y \in V_1, t \in V_2\} \cup \{[x, y, t] \mid d(x, y) = l_3, l_1 - l_2 + 4k; x, y \in V_1, t \in V_2\}$. Counting the number of 2-paths completes the proof. $\square$

We define triplets L_j:

$$L_j = (l_{j1}, l_{j2}, l_{j3}) = \begin{cases} (j, 2k + 1 - j, j) & \text{for } j = 1 \\ (j, 2k + 1 - j, 2k + 1 - j) & \text{for } 2 \leqslant j \leqslant k \\ (j + k, 5k - j, j - k + 1) & \text{for } k + 1 \leqslant j \leqslant 2k - 1. \end{cases}$$

And we define 5-paths P_j, Q_j, R_j $(1 \leqslant j \leqslant 2k - 1)$:

$$P_j = P_{L_j}, \quad Q_j = Q_{L_j}, \quad R_j = R_{L_j}.$$

To show P_j, Q_j, R_j are 5-paths, we will show $l_{j1} + l_{j3} \neq 4k$ and $l_{j2} + l_{j3} \neq 4k \pmod{8k}$ for $1 \leqslant j \leqslant 2k - 1$. For $1 \leqslant j \leqslant k$, we have $l_{j1} + l_{j3} \neq 4k$ and $l_{j2} + l_{j3} \neq 4k \pmod{8k}$ since $1 \leqslant l_{j1}, l_{j2} \leqslant 2k$ and $1 \leqslant l_{j3} \leqslant 2k - 1$. For $k + 1 \leqslant j \leqslant 2k - 1$, we have $l_{j1} + l_{j3} \neq 4k$ and $l_{j2} + l_{j3} \neq 4k \pmod{8k}$ since $l_{j1} + l_{j3}$ and $l_{j2} + l_{j3}$ are odd. Therefore P_j, Q_j, R_j are 5-paths.

Put

$$\mathscr{A}_1 = T\Sigma^*\{P_j, Q_j, R_j \mid 1 \leqslant j \leqslant 2k - 1\},$$

then we have the following proposition.

Proposition 3.2.

$$\begin{aligned}\pi(\mathscr{A}_1) \supset\ & (\Pi_1 \backslash \{[x,t,y] \mid d(x,y)=3k,4k;\ \ x,y \in V_1, t \in V_2\}) \\ & \cup (\Pi_2 \backslash \{[x,y,t] \mid d(x,y)=2k,4k;\ \ x,y \in V_1, t \in V_2\}).\end{aligned}$$

Proof. We apply Proposition 3.1. Since $\{l_{j1}, l_{j2} \mid 1 \leqslant j \leqslant 2k-1\} = \{j \mid 1 \leqslant j \leqslant 4k-1\} \backslash \{3k\}$, we have $\pi(\mathscr{A}_1) \supset \Pi_1 \backslash \{[x,t,y] \mid d(x,y)=3k,4k;\ \ x,y \in V_1, t \in V_2\}$.

Since $\{l_{j3} \mid 1 \leqslant j \leqslant 2k-1\} = \{j \mid 1 \leqslant j \leqslant 2k-1\}$ and $\{l_1 - l_2 + 4k \mid 1 \leqslant j \leqslant 2k-1\} = \{j \mid 2k+1 \leqslant j \leqslant 4k-1\}$, we have $\pi(\mathscr{A}_1) \supset \Pi_2 \backslash \{[x,y,t] \mid d(x,y)=4k,2k;\ \ x,y \in V_1, t \in V_2\}$. Therefore the proposition holds. $\square$

Let S denote the 5-path $[4k, 0, a, 3k, 5k, b]$, and put $\mathscr{A}_2 = T\Sigma\{S\}$.

Proposition 3.3.

$$\begin{aligned}\pi(\mathscr{A}_2) \supset\ & \{[x,t,y] \mid d(x,y)=3k;\ \ x,y \in V_1, t \in V_2\} \\ & \cup \{[x,y,t] \mid d(x,y)=2k,4k;\ \ x,y \in V_1, t \in V_2\}.\end{aligned}$$

Proof. Let $x, y \in V_1$. The 2-path $[x,a,y]$ with $d(x,y)=3k$ is contained in $\Sigma\{S\}$, so $[x,t,y]$ with $d(x,y)=3k$ and $t \in V_2$ is contained in $T\Sigma\{S\}$. The 3-path $[b,5k,3k,a]$ is contained in S, so $[x,y,t]$ with $d(x,y)=2k$ and $t \in V_2$ is contained in $T\Sigma\{S\}$. The 2-path $[4k,0,a]$ is contained in S, so $[x,y,t]$ with $d(x,y)=4k$ and $t \in V_2$ is contained in $T\Sigma\{S\}$. Therefore the proposition holds. $\square$

Put $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$. We call a 5-path in $\mathscr{A}$ a 5-path of type A. We have $|\mathscr{A}| = |\mathscr{A}_1| + |\mathscr{A}_2| = 108k(2k-1) + 72k = 36k(6k-1)$.

Proposition 3.4.

$$\pi(\mathscr{A}) \supset \Pi_1 \cup \Pi_2 \backslash \{[x,t,y] \mid d(x,y)=4k;\ \ x,y \in V_1, t \in V_2\}.$$

Proof. This proposition follows from Propositions 3.2 and 3.3. $\square$

4. 5-paths of type B

We define 5-paths $U_j, V_j, W_j, X_j, Y_j, Z_j$ as follows:

(1) $U_0 = [a, i, 0, e, 4k, f]$, $U_j = \sigma^j(U_0)$ $(0 \leqslant j \leqslant k-1)$,
$U_k = [i, a, k, e, 5k, f]$, $U_{k+j} = \sigma^j(U_k)$ $(0 \leqslant j \leqslant k-1)$,
$U_{2k} = [h, d, 2k, e, 6k, f]$, $U_{2k+j} = \sigma^j(U_{2k})$ $(0 \leqslant j \leqslant 2k-1)$.

(2) $V_0 = [0, b, a, 4k, d, h]$, $V_j = \sigma^j(V_0)$ $(0 \leqslant j \leqslant 4k-1)$.

(3) $W_0 = [h, 0, f, c, 2k, d]$, $W_j = \sigma^j(W_0)$ $(0 \leqslant j \leqslant 2k-1)$,
$W_{2k} = [4k, c, f, 2k, h, d]$, $W_{2k+j} = \sigma^j(W_{2k})$ $(0 \leqslant j \leqslant 2k-1)$.

(4) $X_0 = [d, h, 4k, f, c, 6k]$, $X_j = \sigma^j(X_0)$ $(0 \leqslant j \leqslant 4k-1)$.
(5) $Y_0 = [b, a, 0, g, e, 4k]$, $Y_j = \sigma^j(Y_0)$ $(0 \leqslant j \leqslant 4k-1)$,
$Y_{4k} = [a, b, 4k, g, e, 0]$, $Y_{4k+j} = \sigma^j(Y_{4k})$ $(0 \leqslant j \leqslant 4k-1)$.
(6) $Z_0 = [g, 0, h, d, k, f]$, $Z_j = \sigma^j(Z_0)$ $(0 \leqslant j \leqslant k-1)$.

Put $\mathscr{U} = \{U_j \mid 0 \leqslant j \leqslant 4k-1\}$, $\mathscr{V} = \{V_j \mid 0 \leqslant j \leqslant 4k-1\}$, $\mathscr{W} = \{W_j \mid 0 \leqslant j \leqslant 4k-1\}$, $\mathscr{X} = \{X_j \mid 0 \leqslant j \leqslant 4k-1\}$, $\mathscr{Y} = \{Y_j \mid 0 \leqslant j \leqslant 8k-1\}$, $\mathscr{Z} = \{Z_j \mid 0 \leqslant j \leqslant k-1\}$.

Proposition 4.1.

$$\pi(\Lambda\mathscr{U}) \supset \{[x, t, y] \mid d(x, y) = 4k;\ \ x, y \in V_1, t \in V_2\}.$$

Proof. The 2-path $[x, e, y]$ with $d(x, y) = 4k$ and $x, y \in V_1$ is contained in $\mathscr{U}$. For any $t, u \in V_2$, there exists λ_i such that $\lambda_i(t) = u$. Therefore $[x, t, y]$ with $d(x, y) = 4k$, $x, y \in V_1$ and $t \in V_2$ is contained in $\Lambda\mathscr{U}$. □

Put

$$\mathscr{B} = \Lambda(\mathscr{U} \cup \mathscr{V} \cup \mathscr{W} \cup \mathscr{X} \cup \mathscr{Y} \cup \mathscr{Z}).$$

We call a 5-path in $\mathscr{B}$ a 5-path of type B. We have $|\mathscr{B}| = 225k$.

Proposition 4.2.

$$\pi(\mathscr{B}) \supset \Pi_3 \cup \Pi_4 \cup \{[x, t, y] \mid d(x, y) = 4k;\ \ x, y \in V_1, t \in V_2\}.$$

Proof. Let t, u $(t \neq u)$ be any two elements of V_2. Then t and u belong to the same row, column, right diagonal or left diagonal of A. We will be applying Proposition 2.3 repeatedly.

We first show that $\pi(\mathscr{B}) \supset \Pi_3$. Assume t and u belong to the same column. The 2-path $[a, x, g]$ $(0 \leqslant x \leqslant 4k-1)$ is contained in $\mathscr{Y}$. Since we have $\lambda_j(\{a, g\}) = \{t, u\}$ for some j $(0 \leqslant j \leqslant 8)$ from Proposition 2.3, the 2-path $[t, x, u]$ $(0 \leqslant x \leqslant 4k-1)$ is contained in $\Lambda\mathscr{Y}$. Similarly, the 2-path $[a, x, d]$ $(4k \leqslant x \leqslant 8k-1)$ is contained in $\mathscr{V}$, so the 2-path $[t, x, u]$ $(4k \leqslant x \leqslant 8k-1)$ is contained in $\Lambda\mathscr{Y}$. Therefore the 2-path $[t, x, u]$ $(x \in V_1)$ is contained in $\mathscr{B}$.

Assume t and u belong to the same left diagonal. The 2-path $[h, x, f]$ $(0 \leqslant x \leqslant 4k-1)$ is contained in $\mathscr{W}$ and $[h, x, f]$ $(4k \leqslant x \leqslant 8k-1)$ is contained in $\mathscr{X}$. Therefore the 2-path $[t, x, u]$ $(x \in V_1)$ is contained in $\mathscr{B}$.

Assume t and u belong to the same right diagonal. The 2-path $[e, x, i]$ $(0 \leqslant x \leqslant k-1)$ is contained in $\mathscr{U}$, $[e, x, a]$ $(k \leqslant x \leqslant 2k-1)$ is contained in $\mathscr{U}$, $[c, x, d]$ $(2k \leqslant x \leqslant 4k-1)$ is contained in $\mathscr{W}$ and $[b, x, g]$ $(4k \leqslant x \leqslant 8k-1)$ is contained in $\mathscr{Y}$. Therefore the 2-path $[t, x, u]$ $(x \in V_1)$ is contained in $\mathscr{B}$.

Assume t and u belong to the same row. The 2-path $[g, x, h]$ $(0 \leqslant x \leqslant k-1)$ is contained in $\mathscr{Z}$, $[d, x, f]$ $(k \leqslant x \leqslant 2k-1)$ is contained in $\mathscr{Z}$, $[e, x, d]$ $(2k \leqslant x \leqslant 4k-1)$ is contained in $\mathscr{U}$ and $[e, x, f]$ $(4k \leqslant x \leqslant 8k-1)$ is contained in $\mathscr{U}$. Therefore the 2-path $[t, x, u]$ $(x \in V_1)$ is contained in $\mathscr{B}$. Hence we have $\pi(\mathscr{B}) \supset \Pi_3$.

We now show that $\pi(\mathscr{B}) \supset \Pi_4$. Assume t and u belong to the same left diagonal. The 2-path $[e, g, x]$ $(0 \leqslant x \leqslant 8k-1)$ is contained in $\mathscr{Y}$ and $[g, e, x]$ $(0 \leqslant x \leqslant 8k-1)$ is contained in $\mathscr{Y}$. Therefore the 2-paths $[t, u, x]$ and $[u, t, x]$ $(x \in V_1)$ are contained in $\Lambda\mathscr{Y}$.

Assume t and u belong to the same row. The 2-path $[a, b, x]$ $(0 \leqslant x \leqslant 4k-1)$ is contained in $\mathscr{V}$, $[a, b, x]$ $(4k \leqslant x \leqslant 8k-1)$ is contained in $\mathscr{Y}$, $[b, a, x]$ $(0 \leqslant x \leqslant 4k-1)$ is contained in $\mathscr{Y}$ and $[b, a, x]$ $(4k \leqslant x \leqslant 8k-1)$ is contained in $\mathscr{V}$. Therefore the 2-paths $[t, u, x]$ and $[u, t, x]$ $(x \in V_1)$ are contained in $\mathscr{B}$.

Assume t and u belong to the same column. The 2-path $[c, f, x]$ $(0 \leqslant x \leqslant 4k-1)$ is contained in $\mathscr{W}$, $[c, f, x]$ $(4k \leqslant x \leqslant 8k-1)$ is contained in $\mathscr{X}$, $[f, c, x]$ $(2k \leqslant x \leqslant 6k-1)$ is contained in $\mathscr{W}$ and $[f, c, x]$ $(6k \leqslant x \leqslant 8k-1, 0 \leqslant x \leqslant 2k-1)$ is contained in $\mathscr{X}$. Therefore the 2-paths $[t, u, x]$ and $[u, t, x]$ $(x \in V_1)$ are contained in $\mathscr{B}$.

Assume t and u belong to the same right diagonal. The 2-path $[d, h, x]$ $(0 \leqslant x \leqslant k-1)$ is contained in $\mathscr{Z}$, $[i, a, x]$ $(k \leqslant x \leqslant 2k-1)$ is contained in $\mathscr{U}$, $[d, h, x]$ $(2k \leqslant x \leqslant 4k-1)$ is contained in $\mathscr{W}$ and $[d, h, x]$ $(4k \leqslant x \leqslant 8k-1)$ is contained in $\mathscr{X}$. Since an ordered pair (i, a) is mapped to an ordered pair (d, h) under λ_7, the 2-path $[d, h, x]$ $(x \in V_1)$ is contained in $\mathscr{B}$.

The 2-path $[a, i, x]$ $(0 \leqslant x \leqslant k-1)$ is contained in $\mathscr{U}$, $[h, d, x]$ $(k \leqslant x \leqslant 2k-1)$ is contained in $\mathscr{Z}$, $[h, d, x]$ $(2k \leqslant x \leqslant 4k-1)$ is contained in $\mathscr{U}$ and $[h, d, x]$ $(4k \leqslant x \leqslant 8k-1)$ is contained in $\mathscr{V}$. So the 2-path $[h, d, x]$ $(x \in V_1)$ is contained in $\mathscr{B}$. Therefore the 2-paths $[t, u, x]$ and $[u, t, x]$ $(x \in V_1)$ are contained in $\mathscr{B}$. Hence we have $\pi(\mathscr{B}) \supset \Pi_4$.

From $\pi(\mathscr{B}) \supset \Pi_3$, $\pi(\mathscr{B}) \supset \Pi_4$ and Proposition 4.1, the proposition holds. □

5. A proof of Theorem 1.1

Using the same notation defined in the previous sections, we have the following proposition.

Proposition 5.1. *Let $k \geqslant 1$. Then $\mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$ is a uniform covering of 2-paths with 5-paths in K_{8k+9}.*

Proof. $\mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$ is a set of 5-paths in K_{8k+9}. We have $\pi(\mathscr{A} \cup \mathscr{B}) \supset \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$ from Propositions 3.4 and 4.2. It is trivial that $\pi(\mathscr{A} \cup \mathscr{B}) \subset \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$, so we have $\pi(\mathscr{A} \cup \mathscr{B}) = \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$. Since $|\mathscr{A} \cup \mathscr{B}| = |\mathscr{A}| + |\mathscr{B}| = 27k(8k+7)$ and $|\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4| = 108(8k+7)$, each 2-path in $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$ appears exactly once in $\mathscr{A} \cup \mathscr{B}$. Therefore the proposition holds. □

Finally, we prove Theorem 1.1. Let $n \geqslant 6$. Assume there is a uniform covering $\mathscr{C}$ of 2-paths with 5-paths in K_n. Since there are $n(n-1)(n-2)/2$ 2-paths in K_n and 4 2-paths in a 5-path, we have $|\mathscr{C}| = n(n-1)(n-2)/8$. Thus $n(n-1)(n-2)$ is divisible by 8. Therefore n is even or $n \equiv 1 \pmod 8$.

Conversely, assume n is even or $n \equiv 1 \pmod 8$. When n is even, there is a uniform covering of 2-paths with 5-paths in K_n from Proposition 2.2. When $n = 9$, there is a uniform covering in K_9 from Proposition 2.1. When $n \equiv 1 \pmod 8$ and $n \geqslant 17$, there is a uniform covering in K_n from Proposition 5.1.

This completes the proof of Theorem 1.1. □

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